Exam. Code : 211002
Subject Code : 5542

M.Sc. (Mathematics) 2nd Semester

ALGEBRA–II

Paper–MATH–563

Time Allowed—3 Hours] [Maximum Marks—100

Note :– The candidates are required to attempt **two** questions from each unit. Each question carries equal marks.

UNIT–I

1. (a) Prove that an irreducible element in a PID is always prime. 5

   (b) Prove that every Euclidean Domain is a PID. 5

2. (a) Prove that F[x], F field, is an Euclidean ring. 5

   (b) If R is an integral domain with unit element, then prove that any unit in R[x] must be unit in R. 5

3. (a) Is $\mathbb{Z}[x]$ a Principal Ideal Domain ? Justify your answer. 5

   (b) Prove in UFD, two non-zero elements possess HCF. 5

4. Prove that a if a ring R is PID then it is UFD. Is the converse true ? Justify. 10

UNIT–II

5. (a) If a, b $\in$ K are algebraic over F such that $[F(a) : F] = m$ and $[F(b) : F] = n$ and $\gcd(m, n) = 1$, then prove that $[F(a, b) : F] = mn$. 5

   (b) Prove that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$ is a simple extension. 5

6. (a) Give an example of an algebraic extension of a field which is not finite. 5

   (b) For a prime p, find the degree of splitting field of $x^p - 1$ over $\mathbb{Q}$. 5

7. (a) If K is algebraic over E and E is algebraic over F, then prove that K is algebraic over F. 5

   (b) Prove that the splitting field of a polynomial over F is unique upto F-isomorphism. 5

8. (a) Let θ be a root of an irreducible polynomial $x^3 - 2x - 2$. Then find $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$. 5

   (b) Let f(x) be a non-constant polynomial over field F, then prove that there exists an extension E of F in which f(x) has a root. 5

UNIT–III

9. Prove that a regular n-gon is constructible if and only if $\varphi(n)$ is a power of 2. 10

10. (a) Prove that the characteristic of a finite field F is prime number say p and F contains a subfield isomorphic to $\mathbb{Z}_p$. 5

    (b) Construct a field with 27 elements. 5

11. (a) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. 5

(b) Show that all the roots of an irreducible polynomial over finite field are distinct. 5

12. Prove that a finite separable extension of a field is simple. 10

UNIT–IV

13. Prove that for a Galois extension E/F, there is 1-1 correspondence between the subgroups of G(E/F) and the subfields of E containing F. 10

14. Let $E = \mathbb{Q}\left(\sqrt[3]{2}, w\right)$, where $w^3 = 1, w \neq 1$. Let $H \subseteq G(E/\mathbb{Q})$ and $H = \{\sigma_1, \sigma_2)$ where σ_1 is the identity map and $\sigma_2\left(\sqrt[3]{2}\right) = \sqrt[3]{2}w$ and $\sigma_2(w) = w^2$. Find E_H. 10

15. Suppose that the Galois group G(E/F) of a polynomial f(x) over F is a solvable group, prove that E is solvable by radicals over F. 10

16. (a) Give an example each of a polynomial which is solvable by radicals and a polynomial which is not solvable by radicals. 7

(b) Express $x_1^3 + x_2^3 + x_3^3$ as a rational function of elementary symmetric function. 3

UNIT–V

17. State fundamental theorem of finitely generated module over PID. Prove that a finitely generated torsion–free module over PID is free. 10

18. State and prove Schur's lemma for simple modules. 10

19. Prove that over PID, a submodule of finitely generated module is finitely generated. Is the result true in general ? Justify. 10

20. Let R be a commutative ring with unity and M, N free R-modules. Prove that $Hom_R(M, N)$ is a free R-module if M is finitely generated. Further if N is also finitely generated, then find the basis of $Hom_R(M, N)$. 10